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Discrete Dynamics and Metastability: Mean First Passage Times and Escape Rates

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The problem of escape from a domain of attraction is applied to the case of discrete dynamical systems possessing stable and unstable fixed points. In the presence of noise, the otherwise stable fixed point of a nonlinear map becomes metastable, due to noise-induced hopping events, which eventually pass the unstable fixed point. Exact integral equations for the moments of the first passage time variable are derived, as well as an upper bound for the first moment. In the limit of weak noise, the integral equation for the first moment, i.e., the mean first passage time (MFPT), is treated both numerically and analytically. The exponential leading part of the MFPT is given by the ratio of the noise-induced invariant probability at the stable fixed point and unstable fixed point, respectively. The evaluation of the prefactor is more subtle: It is characterized by a jump at the exit boundaries, which is the result of a discontinuous boundary layer function obeying an inhomogeneous integral equation. The jump at the boundary is shown to be always less than one-half of the maximum value of the MFPT. On the basis of a clear-cut separation of time scales, the MFPT is related to the escape rate to leave the domain of attraction and other transport coefficients, such as the diffusion coefficient. Alternatively, the rate can also be obtained if one evaluates the current-carrying flux that results if particles are continuously injected into the domain of attraction and captured beyond the exit boundaries. The two methods are shown to yield identical results for the escape rate of the weak noise result for the MFPT, respectively. As a byproduct of this study, we obtain general analytic expressions for the invariant probability of noisy maps with a small amount of nonlinearity.

KEY WORDS: Discrete dynamics; circle maps; noise; mean first passage times; metastability; escape rates.

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1. INTRODUCTION

The problem of escape from a locally stable attractor is ubiquitous in the natural sciences, most notably in the chemical kinetics of gases and condensed phases and transport in nonlinear systems such as semiconductors or biological systems, to name but a few. A common situation in many dynamical systems is the presence of several states of local stability, with transitions among these states being induced by random forces. As a matter of fact, a recent issue of this journal was devoted solely to this very issue of escape from metastable states,[1] wherein the present state of the experimental and theoretical achievements and developments are reviewed.

Nonlinear, dissipative, higher dimensional systems are known to possess many simultaneously coexisting basins of attraction. In continuous time, these systems are described by a coupled set of first-order differential equations

$$\dot{x}_\alpha = F_\alpha(x, \lambda) + \xi_\alpha(t), \qquad \alpha = 1,..., d \tag{1.1}$$

$F_\alpha(x, \lambda)$ denotes the deterministic flow of the component x_α; λ is a set of contral parameters, and $\xi_\alpha(t)$ describes the effect of a random environment. In general, analytic solutions of (1.1) are not accessible. In particular, the problem of escape is generally too difficult to be studied analytically on the level of a nonlinear, multidimensional flow.[2,3] Recent experience, however, has shown that, at least in the case of strongly dissipative systems, many characteristic features of a complex nonlinear dynamic system can be extracted from a stroboscope-like discretization in time of a single dynamic trajectory $x_\alpha(t)$. That is, in many cases it is sufficient to focus on one of the variables $x_\alpha(t)$ at successive intervals $\{x_\alpha(t_0),\ x_\alpha(t_0+\tau),\ x_\alpha(t_0+2\tau),...\}$. This yields an *approximation* to the complex flow in (1.1), which takes on the form of a one-dimensional, *discrete*, *noisy dynamics*,[4-7]

$$x_{n+1} = f(x_n, \lambda) + \xi_n \tag{1.2}$$

The predictions of such discrete map dynamics have proven to be not only qualitatively correct, but often even quantitative.[5-7] The recent flood of research on chaos in nonlinear, deterministic, discrete systems (e.g., see Refs. 6 and 7 for reviews) shows that the phase space of even the simplest nonlinear system may contain several basins of attraction, i.e., disjoint regions in phase space, each composed of all the points from which the attractor can be reached upon elapsing time by the deterministic flow. For example, in three dimensions the forced Duffing oscillator

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\gamma x_2 + x_1 - 4x_1^3 + A\cos x_3 \\
\dot{x}_3 &= \omega
\end{aligned} \tag{1.3}$$

shows for the same set of control parameters (A, ω) two locally stable cycles; i.e., it exhibits bistable behavior. Indeed, it is well known that the amplitude response curve $X = (x_1^2 + x_2^2)^{1/2}$ versus frequency ω shows a characteristic *bend* (Ref. 9, Fig. 32); i.e., one observes a hysteresis phenomenon.

The case of the coexistence of stable (or unstable) attractors can be destabilized by a change in control parameter values; thereby one merges two independent attractors into a single one via transfer through an intermediate region which is visited only sporadically. This is the situation of *deterministic diffusion*,[10–12] where the infrequent deterministic hopping leads to a low-frequency tail of the spectral density (Ref. 13, in particular Fig. 1a; also see Refs. 14–18). Alternatively, when the above coexistence is stable, application of noise of internal or of externally imposed origin may induce transitions between two otherwise disjoint regions of phase space. Noise-induced hopping between different attractors has been investigated in the Duffing oscillator via analog simulations in Ref. 17 and via digital simulations in Ref. 13.

Escape times have been estimated in Ref. 19 near the point where an attractor collides with an unstable stationary point, thereby losing its stability (so-called crises).[18] In this situation of deterministic dynamics (absence of random perturbations) the invariant distribution is strongly peaked at the edges of the basin of attraction. Thus, the escape times are almost entirely determined by the statistics of *single jumps*, being induced by externally applied small noise. Series of two or more jumps, which finally drive the system out of the basin of attraction, can be neglected in this very situation near a crisis. In this work, we rather focus on an opposite situation in which single jumps out of the basin of attraction yield only a negligible contribution to the rate.

The emphasis of this work will be on noise-induced escape rates and mean first passage times (MFPT) from a stable fixed point of a discrete map. In particular, our focus will center on noisy circle maps

$$x_{n+1} = x_n + \Omega + a \sin(2\pi x_n) + \xi_n \tag{1.4}$$

The statistical properties of deterministic circle maps have been widely reported in the literature.[11–24] Equation (1.4) without the noise term is a popular mathematical model describing the deterministic behavior of periodically forced, nonlinear oscillator systems, such as occurs for the driven pendulum, phase-locked loops, or ac-driven Josephson junctions.[11–17,19–24] The dynamics of the map in (1.4) is extremely rich, exhibiting bistability, deterministic diffusion, cascading period-doubling bifurcations, phase locking, and chaos.[11–17,19–24] Even when $\Omega = 0$, the phase diagram is still rich (see Fig. 2 in Ref. 11 or Fig. 11 in Ref. 12), possessing limit cycles, period-doubling, deterministic diffusion, and running solutions, if $a > 1/(2\pi)$. For all $a > 0$, the cell boundaries $x = 0$, ± 1, ± 2,... are unstable fixed points. The cell midpoints $\hat{x} = \frac{1}{2} \pm n$ are stable fixed points for $a \in (0, 1/\pi)$, and they become superstable at $a = 1/(2\pi)$; a bifurcation to a period-2 solution occurs[11] for the control parameter value $a = 1/\pi$. In the following we shall restrict the discussion to the regime $a \in (0, 1/(2\pi))$; i.e., for zero external driving, $\Omega = 0$, the noisy discrete dynamics is given by

$$x_{n+1} = f(x_n) + \xi_n \equiv x_n + a \sin 2\pi x_n + \xi_n, \qquad 0 < a < 1/(2\pi) \tag{1.5}$$

The random forces $\{\xi_n\}$ are assumed to be independent and identically distributed according to a probability density $\rho(\xi)$,

$$P[\xi_n \in (\xi, \xi + d\xi)] = \rho(\xi)\, d\xi \tag{1.6}$$

with a vanishing mean $\langle \xi_n \rangle = 0$. Taking the stable fixed point $x_0 = 1/2$ as the point of reference, our focus is on the noise-induced escape from the domain $I = [0, 1]$ of attraction of x_0. With an unbounded random force ξ_n, i.e., $\rho(\xi) > 0$ for all finite ξ, the escape from the stable fixed point x_0 will occur with certainty, even for arbitrarily small noise strength $\langle \xi_n^2 \rangle > 0$. By contrast, for a bounded noise source, i.e., $\rho(\xi) = 0$ for all $|\xi| > \xi_0$, this need not be the case. For example, if $0 < \xi_0 < 1/a$, the noise-limited map functions (Ref. 25, in particular Section 3.1)

$$x_{n+1} = x_n + a \sin 2\pi x_n \pm \xi_0 \tag{1.7}$$

possess shifted stable fixed points $x_\pm$ within $I = [0, 1]$. As a consequence, the stationary probability of x_n in the presence of noise is nonvanishing only on the subinterval $(x_-, x_+) \in I$, and zero outside. Therefore, an escape from x_0 is impossible.

2. NOISE-INDUCED INVARIANT PROBABILITY: CASE OF SMALL NONLINEARITY

Let us consider the map dynamics (1.5) with ξ_n being an independent, identically distributed random perturbation of *unbounded* support. Throughout this paper we shall restrict the discussion to additive noise ξ_n. A generalization of our results for multiplicative noise

$$\xi_n \to g(x_n)\, \xi_n \tag{2.1}$$

is straightforward, but will not be implemented here, for the sake of clarity. The distribution $\rho(\xi)$ of the unbounded random variable can be arbitrary.

For our explicit calculations, however, we use a Gaussian of vanishing mean

$$\rho(\xi) = \frac{1}{(2\pi\varepsilon)^{1/2}} \exp - \frac{\xi^2}{2\varepsilon} \tag{2.2}$$

Due to the independence of ξ_n and ξ_m, $(n \neq m)$, the process x_n, (1.5), is a Markov process. Let

$$\begin{aligned} P(x \mid y) &= \int_{-\infty}^{\infty} d\xi\, \delta(x - f(y) - \xi)\, \rho(\xi) \\ &= \rho(x - f(y)) \\ &= \frac{1}{(2\pi\varepsilon)^{1/2}} \exp - \frac{[x - f(y)]^2}{2\varepsilon} \end{aligned} \tag{2.3}$$

denote the transition probability to go from $x_n = y$ to $x_{n+1} = x$ in a single step. Then, the probability $W_n(x)$ to find x in the interval $(x, x + dx)$ satisfies the master equation[26,27]

$$W_{n+1}(x) = \int_{-\infty}^{\infty} dy\, P(x \mid y)\, W_n(y) \tag{2.4}$$

More generally, one obtains for the conditional probability $P_n(x \mid y)$ to get in n steps from y into $(x, x \pm dx)$, in virtue of (2.3), (2.4), the *forward* equation

$$P_{n+1}(x \mid y) = \int dz\, P(x \mid z)\, P_n(z \mid y), \qquad P_0(z \mid y) = \delta(z - y) \tag{2.5}$$

Likewise, one finds from taking the adjoint in (2.5) the *backward* equation

$$P_{n+1}(y \mid x) = \int dz\, P_n(y \mid z)\, P(z \mid x) \tag{2.6}$$

Before we investigate in more detail the invariant probability $W(x)$, i.e., the eigenvalue $(\mu_n = 1)$ solution of (2.4),

$$W(x) = \int_{-\infty}^{\infty} dy\, P(x \mid y)\, W(y) \tag{2.7}$$

we define a class of map functions $\{f(x)\}$ [see (1.2)] with the characteristic property that $f(x)$ deviates only weakly from the identity map. Specifically, we consider the class

$$f(x) \equiv x - a \frac{dU(x)}{dx}, \qquad a \ll 1 \tag{2.8}$$

where $a>0$ is small and $U(x)$ is an even, bounded function such that $x_0=1/2$ is a stable fixed point and the cell boundaries $x_-=0$, $x_+=1$ are unstable fixed points. Clearly, Eq. (1.5) with $a<1/(2\pi)$ is a member of (2.8) with

$$U(x)=\frac{1}{2\pi}\cos(2\pi x) \tag{2.9}$$

Typical members of the set of functions $\{f(x)\}$, (2.8), are sketched in Fig. 1. If we combine (2.7) with (2.3) and (2.2), the invariant probability $W(x)$ for the class (2.8) explicitly reads

$$W(x)=\frac{1}{(2\pi\varepsilon)^{1/2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\varepsilon}\left[x-y+aU'(y)\right]^2\right\}W(y)\,dy \tag{2.10}$$

where $U'(y)$ denotes the derivative $dU(y)/dy$. Generally, (2.10) cannot be solved exactly. An exception presents the linear map dynamics with $U(x)=\frac{1}{2}\omega^2x^2$. This yields the exact result

$$W(x)^{\text{linear}}=Z^{-1}\exp[-(2a\omega^2-a^2\omega^4)x^2/(2\varepsilon)] \tag{2.11}$$

With a small, (2.10) can be solved approximately. Setting

$$u\equiv(x-y+aU'(y)/(2\varepsilon)^{1/2}$$

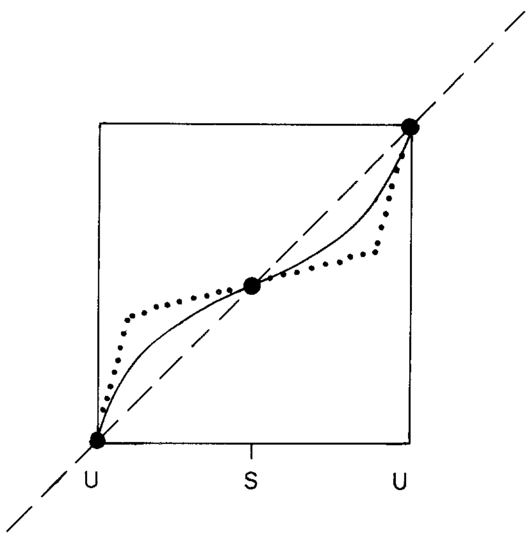

Fig. 1. Typical members of the map functions (2.9) exhibit one stable (s) fixed point and two unstable (u) fixed points at the cell boundaries.

we find

$$y = x - (2\varepsilon)^{1/2} u + aU'[x - (2\varepsilon)^{1/2} u] + O(a^2) \tag{2.12}$$

A substitution of (2.12) into (2.10) yields

$$\begin{aligned} W(x) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{du}{1 - aU''[x - (2\varepsilon)^{1/2} u + O(a^2)]} \\ &\quad \times \exp(-u^2)\, W\{x - (2\varepsilon)^{1/2} u + U'[x - (2\varepsilon)^{1/2} u] + O(a^2)\} \\ &= W(x) + a(U'W')' + \frac{\varepsilon}{2} W'' + O(a^2) \end{aligned}$$

Integrating the differential form

$$a(U'W')' + \tfrac{1}{2}\varepsilon W'' = 0 \tag{2.13}$$

twice, one obtains for the approximate invariant probability[4]

$$W(x) = Z^{-1} \exp\left[-\frac{2aU(x)}{\varepsilon} + O\left(\frac{a^2}{\varepsilon}\right) \right] \tag{2.14}$$

For the climbing sine map (1.5) we obtain the periodic invariant probability

$$W(x) = Z^{-1} \exp\left[-\frac{a \cos(2\pi x)}{\pi\varepsilon} \right], \qquad a \ll 1 \tag{2.15}$$

This invariant probability enters the result for the MFPT at low noise intensity, to be considered in Section 3.2.

3. DISCRETE DYNAMICS AND METASTABILITY: MEAN FIRST PASSAGE TIME

In this Section, we elaborate on the noise-induced escape of circle maps of the type in (2.8). In particular, we consider the equation obeyed by the MFPT of the random variable $\tau(x)$, the random time to leave the domain of attraction $I = [0, 1]$ for the first time, without ever returning into I, if the random walker started out at $x_{n=0} = x$.[5]

[4] In the derivation of (2.14) we did not make use of the fact that $f(x) = x - aU'(x)$ refers to a metastable situation; only the limit $a \ll 1$ has been utilized. Thus, the approximation in (2.14) holds for more general functions $f(x)$, not restricted to describing noise-induced bistability.

[5] The results of Sections 3 and 4 were presented at the Statphys-16 Conference, Boston, August 10–15, 1986.

3.1. Integral Equations for the Moments of the First Passage Time Variable

To begin, let $W(n|x)$ denote the probability that up to time n a random walker that started out at x in I has *not yet left* the domain of attraction. By use of (2.6), we have

$$W(n|x) = \int_I \hat{P}_n(y|x)\, dy \tag{3.1}$$

where $\hat{P}_n(x|y)$ is the probability to reach x in n steps without having left I. The probability $\hat{P}_n$ obeys the backward equation

$$\hat{P}_{n+1}(x|y) = \int_I \hat{P}_n(x|z)\, \hat{P}(z|y)\, dz \tag{3.2a}$$

where we have introduced

$$\hat{P}_1(x|y) = \hat{P}(x|y) = \begin{cases} P(x|y), & y \text{ in } I \\ 0, & \text{otherwise} \end{cases} \tag{3.2b}$$

$W(n|x)$ in (3.1) is obviously a decreasing function with n increasing; it accounts for the fact that an escape becomes more and more likely with increasing n. On noting that an exit between n and $(n+1)$ occurs with probability $W(n|x) - W(n+1|x)$, we readily obtain the MFPT $t_1(x)$ as

$$\begin{aligned} t_1(x) &= \langle \tau(x) \rangle \\ &= \sum_{n=0}^{\infty} (n+1)[W(n|x) - W(n+1|x)] \\ &= \sum_{n=0}^{\infty} W(n|x) \end{aligned} \tag{3.3}$$

On the other hand, by integrating the backward equation (3.2) over y, summing over all n, one obtains in view of (3.3) the integral equation

$$\begin{aligned} t_1(x) - \chi_I(x) &= \int_I \hat{P}(y|x)\, t_1(y)\, dy \\ &= \int_{-\infty}^{\infty} P(y|x)\, t_1(y)\, dy \end{aligned} \tag{3.4a}$$

where

$$t_1(x) = 0, \qquad x \text{ outside } I = [0, 1]$$

and

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \text{ in } I \\ 0 & \text{otherwise} \end{cases} \tag{3.4b}$$

Thus, the MFPT for a discrete dynamics is governed by an inhomogeneous integral equation. This result, (3.4), was first derived in Ref. 28.

It is also possible to derive explicit integral equations for the second and higher moments of $\tau(x)$; i.e.,

$$t_m(x) = \sum_{n=0}^{\infty} (n+1)^m [W(n \mid x) - W(n+1 \mid x)], \qquad m \geqslant 1 \tag{3.5}$$

For example, the second moment obeys

$$t_2(x) - 2t_1(x) + \chi_I(x) = \int_I \hat{P}(y \mid x)\, t_2(y)\, dy \tag{3.6}$$

Exact solutions of (3.4) and (3.6) are not possible, in general. No solutions of (3.4) or (3.6) have been found. For the class of map functions in (2.8) one has for x in I from (2.3) and (3.4) the integral equation

$$t_1(x) - 1 = \int_0^1 \rho(y - x + aU'(x))\, t_1(y)\, dy \tag{3.7}$$

Equation (3.7) has a unique solution if

$$\int_0^1 \rho(z)\, dz < 1$$

In this case one can show that the homogeneous equation in (3.7) only possesses the trivial solution $t_1(x) = 0$. Moreover, with 0 and 1 both exit points for $\tau(x)$, $t_1(x)$ attains its maximal value at $x_0 = \frac{1}{2}$. Setting $t_1(1/2) = T$, it follows from (3.7) that

$$\begin{aligned} T - 1 &= \int_0^1 \rho(y - \tfrac{1}{2})\, t_1(y)\, dy \\ &\leqslant T \int_0^1 \rho(y - \tfrac{1}{2})\, dy \end{aligned}$$

i.e., the MFPT obeys the *inequality*

$$t_1(x) \leqslant T \leqslant \frac{1}{1 - \int_0^1 \rho(y - \frac{1}{2})\, dy} \equiv T_A \tag{3.8}$$

T_A, as the inverse of the probability to leave the interval of attraction in a single step starting from $x_0 = \frac{1}{2}$, represents an *upper bound* for $t_1(x)$. The estimate T_A just coincides with the approximation of Arecchi *et al.*[19] Moreover, in contrast to the usual case of Fokker–Planck processes,[1,2] $t_1(x) = t_1(1-x)$ possesses a jump[6] at x_- (or x_+).[7] Sitting at a boundary point, say $x_- = 0$, the random walker will reenter the interior of the interval (0, 1) with probability $p = \int_0^1 P(y \mid 0)\, dy$. Consequently, the mean first passage time from $x_- = 0$, $t_1(0)$, is larger than the unit time step. An estimate of the jump is obtained from (3.4) at $x_- = 0$,

$$t_1(0) - 1 = \int_0^1 \rho(y)\, t_1(y)\, dy$$

or (3.9)

$$t_1(0) = CT + 1 > 0$$

where, with $t_1(x) = T\tilde{h}(x)$, $\tilde{h}(x) < 1$, and symmetric noise, $\rho(y) = \rho(-y)$

$$C = \int_0^1 \rho(y)\, \tilde{h}(y)\, dy < \frac{1}{2} \tag{3.10}$$

Figure 2 depicts the qualitative behavior of the MFPT $t_1(x)$. Clearly, all of the characteristic difficulties, such as boundary jumps and absorptive lines

[6] Similar jumps for the MFPT occur for continuous-time processes driven by non-Gaussian noise sources; see Refs. 29–32.

[7] Note that for a Fokker–Planck process the continuity of the mean first passage time is due to the highly irregular behavior of the trajectory of a Wiener process: such a trajectory crosses a certain level infinitely many times within an arbitrarily short amount of time, once the first crossing has happened.

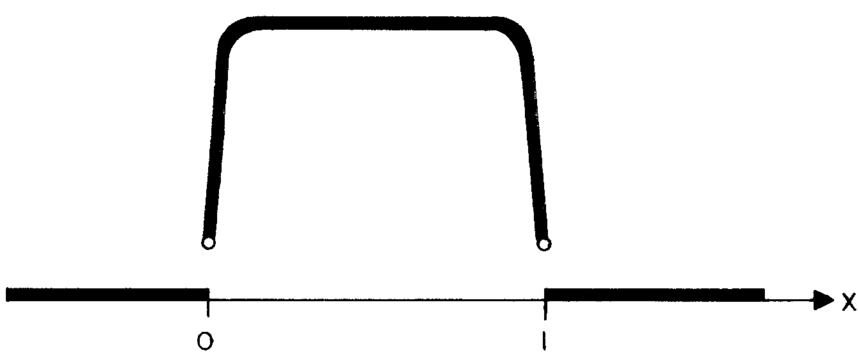

Fig. 2. Qualitative sketch of the MFPT $t(x)$, (3.4), versus x. Illustrated are the absorbing lines outside the domain of attraction, as well as the typical boundary jumps at the exit points.

$(-\infty, 0^-)$ and $(1^+, \infty)$ (see Fig. 2), are inherent in the integral operator structure (3.4).

3.2. The Weak Noise Analysis of the MFPT

At a weak noise level, $\langle \xi^2 \rangle = \varepsilon \ll 1$, the MFPT attains a very large value, of the order of the escape time from the metastable fixed point $x_0 = \frac{1}{2}$. Furthermore, $t_1(x)$ essentially approaches a constant value $t_1(x) \simeq T$, inside the domain of attraction. Deviations of $t_1(x)$ from the constant T primarily occur near the boundaries of the domain of attraction. Therefore, we can set

$$t_1(x) = T\tilde{h}(x), \qquad \tilde{h}(\tfrac{1}{2}) = 1 \tag{3.11}$$

If we insert the ansatz (3.11) into (3.7), we find

$$\tilde{h}(x) - T^{-1} = \int_0^1 \rho(y - x + aU'(x))\, \tilde{h}(y)\, dy \tag{3.12}$$

In view of the characteristic behavior of $t_1(x)$ (see Fig. 2), $\tilde{h}(x)$ varies within a boundary layer region of width $\varepsilon^{1/2}$, and takes on an approximately constant value inside the domain of attraction. The focus in (3.12) is thus on the variation of $\tilde{h}(x)$ near the unstable fixed points $x_- = 0$ and $x_+ = 1$. For example, if we linearize (3.12) within the boundary layer of width $\varepsilon^{1/2}$ around $x_- = 0$, we can write for the scaled boundary layer function $h(x)$,

$$h(x) = \tilde{h}((2\varepsilon)^{1/2} x) \tag{3.13}$$

the integral equation

$$h(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp[-(y - Ax)^2]\, h(y)\, dy \tag{3.14a}$$

Here, we made use of (2.2) for $\rho(\xi)$, and

$$A = 1 - aU''(0) > 1 \tag{3.14b}$$

In addition, we have neglected the small inhomogeneous contribution T^{-1} [see (3.8)] and approximated the upper limit of integration $1/\varepsilon$ by infinity. The asymptotic behavior of the scaled boundary layer function $h(x)$ is normalized such that $h(x) \to 1$ for $x \to \infty$. For the derivative of (3.14) one ends up with the *inhomogeneous* integral equation

$$h'(x) = \frac{A}{\sqrt{\pi}} \exp[-(Ax)^2]\, h(0) + \frac{A}{\sqrt{\pi}} \int_0^\infty dy \exp[-(y - Ax)^2]\, h'(y) \tag{3.15}$$

In the asymptotic regime ($x \to \infty$), (3.15) simplifies to give, with $u = y - Ax$,

$$h'(x) \underset{x \to \infty}{=} \frac{A}{\sqrt{\pi}} \int_{-Ax}^{\infty} du \exp(-u^2)\, h'(u + Ax) \tag{3.16a}$$

If we replace the lower integration limit by $-\infty$, the solution of (3.16a) yields the asymptotic behavior

$$h'(x) = \exp[-(A^2 - 1)x^2], \qquad x \to \infty \tag{3.16b}$$

Applying Nyström's method,[33] we obtain a numerical solution of $h(x)$ conveniently from (3.15). Results for the normalized, scaled boundary layer function $h(x)$ are depicted in Fig. 3.

An explicit expression for the constant part T can be obtained in a similar way as in the Fokker–Planck case[34]: First we multiply (3.7) with the invariant probability $W(x)$, and integrate over all x in $I = [0, 1]$; i.e.,

$$\begin{aligned}
&\int_0^1 t_1(x)\, W(x)\, dx - \int_0^1 W(x)\, dx \\
&\quad = \int_0^1 dx \int_0^1 dy\, P(y \mid x)\, t_1(y)\, W(x) \\
&\quad = \int_{-\infty}^{\infty} dx \int_0^1 dy\, P(y \mid x)\, t_1(y)\, W(x) \\
&\qquad - \left(\int_{-\infty}^{0} + \int_1^{\infty} \right) dx \int_0^1 dy\, P(y \mid x)\, t_1(y)\, W(x) \\
&\quad = \int_0^1 dy\, t_1(y)\, W(y) - \left(\int_{-\infty}^{0} + \int_1^{\infty} \right) dx \int_0^1 dy\, P(y \mid x)\, t_1(y)\, W(x)
\end{aligned} \tag{3.17}$$

In the last step, we made use of the invariant property (2.7). With (3.11) we thus find the central result

$$T^{-1} = \frac{\left(\int_{-\infty}^{0} + \int_1^{\infty}\right) dx\, W(x) \int_0^1 dy\, P(y \mid x)\, \tilde{h}(y)}{\int_0^1 W(x)\, dx} \equiv T_-^{-1} + T_+^{-1} \tag{3.18}$$

Here, T_- refers to the escape time via the left exit point $x_- = 0$, being determined by the $(\int_{-\infty}^{0} dx \cdots)$ contribution, and T_+ is defined correspondingly. The result (3.18) is an *exact* expression for $t(\frac{1}{2}) = T$. At weak noise, $\varepsilon \ll 1$, (3.18) can be simplified further: At the fixed points x_-, x_0, and x_+, $W(x)$ is sharply peaked. Moreover, if $\hat{x}$ denotes a fixed point

$$f(\hat{x} + \eta) = \hat{x} + \eta f'(\hat{x}) = \hat{x} + [1 - aU''(\hat{x})]\eta$$

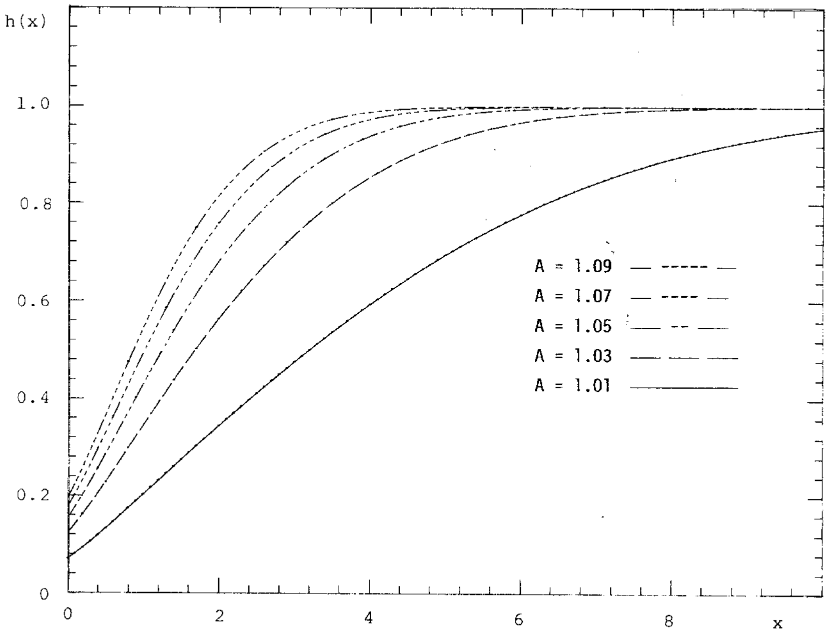

Fig. 3. Numerical solution of the scaled boundary layer function (3.14), (3.15) for various A values [see (3.14b)].

the Gaussian approximation to (2.10) reads

$$W(x) \simeq W(\hat{x}) \exp(\{[1 - aU''(\hat{x})]^2 - 1\}(\hat{x} - x)^2/(2\varepsilon)), \qquad x \simeq \hat{x} \tag{3.19}$$

With a steepest descent approximation the denominator in (3.18) becomes

$$\int_0^1 W(x)\,dx \simeq \left(\frac{2\pi\varepsilon}{1 - [1 - aU''(\frac{1}{2})]^2}\right)^{1/2} W\left(\frac{1}{2}\right) \tag{3.20a}$$

For a small nonlinearity a, this simplifies further [see (2.14)] to give

$$\int_0^1 W(x)\,dx \simeq \left(\frac{\pi\varepsilon}{aU''(\frac{1}{2})}\right)^{1/2} \exp -\frac{2aU(\frac{1}{2})}{\varepsilon} \tag{3.20b}$$

The numerator simplifies as well. In the same spirit as above, we can write

$$\int_{-\infty}^0 dx\, W(x) \int_0^1 dy\, P(y \mid x)\, \tilde{h}(y)$$
$$\simeq W(0)(2\pi\varepsilon)^{-1/2} \int_{-\infty}^0 dx \exp(\{[1 - aU''(0)]^2 - 1\}x^2/(2\varepsilon))$$
$$\times \int_0^1 dy\; \tilde{h}(y) \exp\{-[y - x + aU'(x)]^2/(2\varepsilon)\} \tag{3.21}$$

Upon combining (3.21) with (3.20), we find for T_- the expression

$$T_- = (2\pi\varepsilon)^{1/2}\{1-[1-aU''(x_0)]^2\}^{-1/2}$$
$$\times\left\{\int_{-\infty}^{0} dx\, \exp[(A^2-1)x^2/(2\varepsilon)]\right.$$
$$\left.\times\int_0^1 dy\, \tilde{h}(y)\exp[-(y-Ax)^2/(2\varepsilon)]\right\}^{-1}$$
$$\times W(\tfrac{1}{2})/W(0) \tag{3.22}$$

A corresponding result holds for T_+. The leading exponential part of the escape time is read off from (3.18) and (3.22) as

$$T \propto \frac{W(\frac{1}{2})}{W(0)+W(1)} \sim \frac{1}{2}\exp\frac{2a}{\pi\varepsilon} \tag{3.23}$$

The expression (3.22) is an appealing result for the MFPT at weak noise. It can be evaluated further as follows: The denominator in (3.22) obeys, in terms of the scaled variable $x \to (2\varepsilon)^{1/2}x$,

$$\int_{-\infty}^{0} dx \int_0^1 dy\, P(y\,|\,x)\,\tilde{h}(y)\, W(x) \to \tfrac{1}{2}\varepsilon^{1/2} W(0)\, R(A), \qquad \varepsilon \ll 1$$

where

$$R(A) = \int_0^\infty dy[1-\phi(Ax)]\, h(y)\exp[(A^2-1)\, y^2] \tag{3.24}$$

Here, we replaced the upper integration limit occurring in (3.24), $1/(2\varepsilon)^{1/2}$, by infinity. $\phi(x)$ denotes the error function. $R(A)$ is an ε-independent function, whose numerical behavior is depicted in Fig. 4. Within the numerical accuracy, the result of $R(A)$ is excellently approximated by (see Fig. 4)

$$R(A) = \left(\frac{A^2-1}{\pi}\right)^{1/2} - \frac{1}{2\sqrt{\pi}}(A^2-1) \tag{3.25}$$

Inserting (2.9), (3.20b), and (3.25) into the result (3.22), one explicitly finds

$$T_- = \frac{1}{2a}\exp\frac{2a}{\pi\varepsilon} = T_+, \qquad \frac{a}{\varepsilon} > 1 \tag{3.26}$$

This is the main result of the weak noise analysis. In obtaining (3.26) we made use of the numerical solution of the boundary layer function $h(x)$, which indirectly enters in (3.25). An approximate, fully analytical treatment for $h(x)$, respectively T_-, will be presented in the following subsection.

In the remainder of this subsection we compare the MFPT at weak noise, (3.22), with closely related transport coefficients. First, there is a relationship with the rate of escape λ. In terms of the forward rate λ^+ and the backward rate λ^- the rate of escape is related to T by

$$\lambda = \lambda^+ + \lambda^- = 1/(2T) \tag{3.27}$$

The factor of one-half takes into account that, in absence of a capture beyond the unstable fixed points $x_- = 0$, $x_+ = 1$, half of the number of

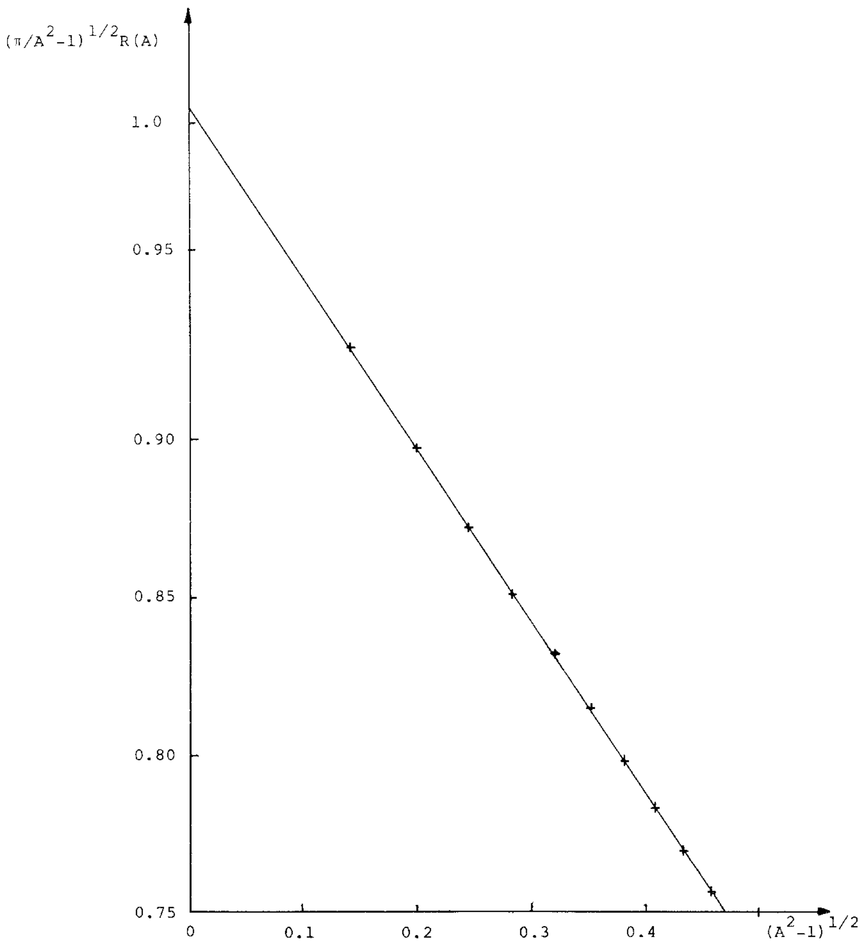

Fig. 4. (—) Scaling function $R(A)$ in (3.25) versus $(A^2-1)^{1/2}$ compared against ($\times$) the numerical results.

particles would return into the domain of attraction.[35] Likewise, λ^+ is given by

$$\lambda^+ = \lambda^- = \tfrac{1}{2}(1/2T) = 1/2T_- \tag{3.28}$$

$2T$ corresponds to the MFPT of a random walker that is allowed to exit at one boundary only. In the absence of any capture, the random walker undergoes a noise-induced diffusive motion with a diffusion coefficient D given by

$$\langle (x_n - \langle x_n \rangle)^2 \rangle \xrightarrow[n \to \infty]{} 2Dn \tag{3.29}$$

D itself is determined via the rates and the step size, $(x_+ - x_-) = 1$, i.e.,

$$D = \tfrac{1}{2}(\lambda^+ + \lambda^-)(x_+ - x_-)^2 = 1/(4T) \tag{3.30}$$

3.3. Approximate Analytical Treatment

Unlike the case of Fokker–Planck processes,[1 3,34] there is no standard method that enables one to obtain the boundary layer function $h(x)$ [(3.14), (3.15)] in closed form. As already mentioned, for weak noise the function $h(x)$ is an almost constant function in the domain of attraction I, apart from a crucial change in a narrow region around the unstable fixed point(s). From (3.14) we see that $h(x)$ depends on the map function $U(x)$ only via $U''(0) < 0$. In the following, we seek to approximate the behavior of $h'(x)$ by its asymptotic behavior (3.16b) for large values of x; i.e., we set

$$h'(x) \sim \exp[-(A^2 - 1)x^2] \tag{3.31}$$

Then, $h(x)$ is readily integrated to give

$$h^{as}(x) = h^{as}(0)\left[1 + \int_0^x h'(y)\,dy\right] \tag{3.32a}$$

with

$$h^{as}(0) = \left[1 + \int_0^\infty h'(y)\,dy\right]^{-1} \tag{3.32b}$$

This extrapolated asymptotic function is compared with the numerical solution of the integral equation in Fig. 5 for a particular value of A. Significant deviations apear for small values of x.

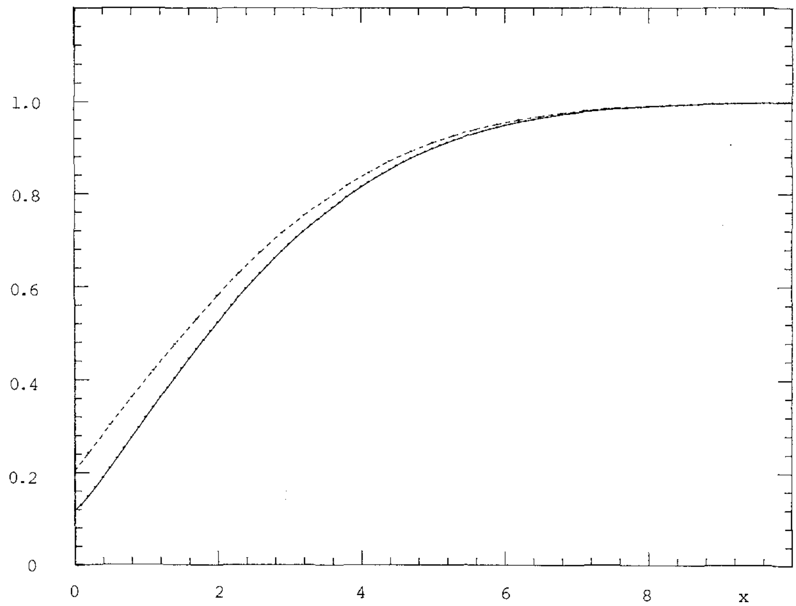

Fig. 5. (—) Numerical solution of the scaled boundary layer function (3.14), $h(x)$, and (– –) the extrapolated asymptotic curve (3.32), $h^{as}(x)$, for the same value of $A = 1.025$.

A straightforward, although somewhat cumbersome evaluation of the integrals in (3.32) and (3.24), involving error functions, then yields for (3.22) the answer

$$T_{-}^{as} = \frac{2^{-1/2}}{2a} \exp \frac{2a}{\pi\varepsilon} \tag{3.33}$$

The sensitivity of the integrals (3.32) and (3.24) with respect to the small-x behavior of $h(x)$ shows up in a different factor $2^{-1/2}$ in (3.33) compared with the exact prefactor in (3.26).

4. DISCRETE DYNAMICS AND METASTABILITY: RATE APPROACH

In the previous section, we presented the method of the MFPT related to the rate by (3.27). The concept of the MFPT, however, is plagued by difficulties, which originate from the integral equation obeyed by the MFPT, (3.4), or the integral equation obeyed by the boundary layer function $h(x)$, (3.14) and (3.15). Although there exists a direct relationship between the MFPT and the rate, there are alternative methods available, which prove to be more suitable if one seeks only the rate itself. Following some ideas pioneered by Kramers,[(36),8] one can express the rate in terms of a net probability current at the exit point that results if one continuously injects particles into the domain of attraction, which are then captured beyond the exist point(s). This perturbation of the system dynamics builds up a stationary nonequilibrium current, carried by a nonequilibrium invariant probability $W_0(x)$. If we consider the backward rate λ^-, we can write

$$\lambda^- = \frac{\text{net probability current at } x_- = 0}{\text{population in the domain of attraction}} = \frac{J_0^-}{n_0} \tag{4.1}$$

The stationary, flux-carrying probability $W_0^-(x)$ obeys

$$W_0^-(x) = \eta^-(x)\, W(x) = \int_{-\infty}^{\infty} P(x \mid y)\, W_0^-(y)\, dy \tag{4.2}$$

The form function $\eta^-(x)$ in (4.2) describes the deviation of the current-carrying probability $W_0^-(x)$ from the zero-current carrying invariant probability $W(x)$, (2.7). Outside the domain of attraction, $W_0^-(x)$ will approach zero rather rapidly, with its characteristic change occurring near the exit boundary. Inside the domain of attraction the equilibrium will be perturbed only slightly. Thus, we expect a qualitative behavior for $\eta^-(x)$ as sketched in Fig. 6.

In terms of $W_0^-(x)$ the current J_0^- reads

$$J_0^- = \int_{-\infty}^{0} dx \int_0^{\infty} P(x \mid y)\, W_0^-(y)\, dy - \int_0^{\infty} dx \int_{-\infty}^{0} P(x \mid y)\, W_0^-(y)\, dy \tag{4.3}$$

[8] See Ref. 37 for a recent review of the rate approach.

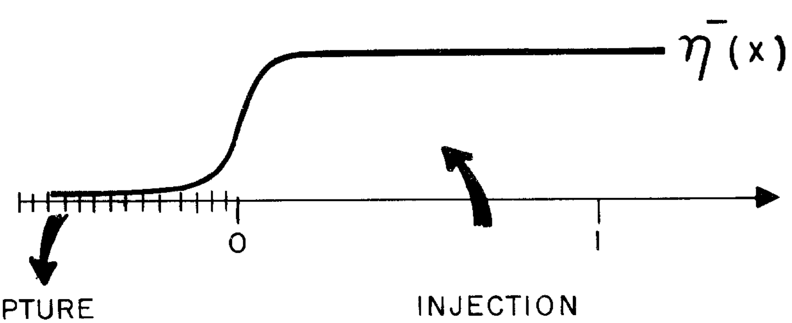

Fig. 6. Typical behavior of the form function $\eta^-(x)$ describing the deviation from the equilibrium invariant probability. It changes drastically within the boundary layer width $(2\varepsilon)^{1/2}$.

The first term describes the flux out of $\hat{I}=(0, \infty)$, while the second term compensates for particles that flow into the domain $\hat{I}$. The population n_0 is

$$n_0=\int_0^1 W_0^-(x)\,dx \tag{4.4a}$$

With $\eta^-(x)\sim 1$ for x in $(0^+, 1^-)$, n_0 can be approximated by

$$n_0\simeq\int_0^1 W(x)\,dx \tag{4.4b}$$

because $W(x)$ attains its minimum around $x=0$, $x=1$, and $\eta^-(x)$ markedly changes from its constant value only within a narrow width around $x=0$. Now we focus on the behavior of $\eta^-(x)$ near $x\simeq 0$: For the invariant probability we have in the limit of weak noise $\varepsilon \ll 1$ [see (3.19)]

$$W(x)\simeq W(0)\exp[(A^2-1)x^2/(2\varepsilon)], \qquad x\simeq 0 \tag{4.5a}$$

and for the transition probability

$$P(x\mid y)\simeq(2\pi\varepsilon)^{-1/2}\exp[-(y-Ax)^2/(2\varepsilon)], \qquad x\simeq 0 \tag{4.5b}$$

Thus, in the neighborhood of $(x\simeq 0,\ y\simeq 0)$, one finds the detailed balance relation

$$P(x\mid y)\,W(y)=P(y\mid x)\,W(x) \tag{4.6}$$

From (4.2), one then finds in the regime around $x_-=0$

$$\eta^-(x)\,W(x)\simeq\int_{-\infty}^{\infty}(2\pi\varepsilon)^{-1/2}\exp[-(y-Ax)^2/(2\varepsilon)]\,\eta^-(y)\,W(y)\,dy \tag{4.7}$$

Utilizing the detailed balance relation in (4.6), one obtains the integral equation

$$\eta^-(x)=\int_{-\infty}^{\infty}(2\pi\varepsilon)^{-1/2}\,\eta^-(y)\exp[-(y-Ax)^2/(2\varepsilon)]\,dy \tag{4.8}$$

The solution of (4.8) with the normalized asymptotic behavior $\eta^-(x)\to 1$, $x\gg 0$, is readily found to read

$$\eta^-(x)=\left(\frac{A^2-1}{2\pi\varepsilon}\right)^{1/2}\int_{-\infty}^{x} dy\exp-\frac{(A^2-1)\,y^2}{2\varepsilon} \tag{4.9}$$

In agreement with the sketch in Fig. 6, $\eta^-(x)$ rapidly approaches 1 for $x>0$, and decreases rapidly for $x<0$.

With (4.9), (4.6), and (4.5) the current J_0^- can be evaluated to yield the weak noise result (see also the Appendix)

$$J_0^- = \tfrac{1}{2}\varepsilon\eta^{-\prime}(0)\, W(0) = [(A^2-1)/2\pi\varepsilon]^{1/2}\tfrac{1}{2}\varepsilon W(0) \qquad (4.10)$$

By use of (3.20b) for the population n_0, (4.4b), the backward rate λ^- reads for the climbing since-map explicitly

$$\lambda^- = \frac{1}{2}\varepsilon\left(\frac{A^2-1}{2\pi\varepsilon}\right)^{1/2}\left[\frac{aU''(\frac{1}{2})}{\pi\varepsilon}\right]^{1/2}\frac{W(0)}{W(\frac{1}{2})} = a\exp-\frac{2a}{\pi\varepsilon} \qquad (4.11)$$

This result equals (3.28) with T_- given by (3.26); i.e., the rate λ^- equals one-half of the inverse mean first passage time T_-. The total rate λ is readily found to be

$$\lambda = 2\lambda^- = 2a\exp(-2a/\pi\varepsilon) \qquad (4.12)$$

Note that the underlying assumptions inherent in (4.11) and (4.12) are based on the steepest descent approximation (3.20), (3.21), and a small nonlinearity, $a < 1/(2\pi)$. This implies that the Arrhenius factor $2a/\pi\varepsilon$ in the rate expressions (4.11) and (4.12) must exceed the order of unity in order to present a meaningful result. Thus, the behavior of the rates in the limit $a \to 0$ with ε kept fixed is not within the regime of validity of (4.11) and (4.12).

5. CONCLUSIONS

In this paper we have considered the escape dynamics of periodic, discrete maps perturbed by noise. In particular, we have evaluated in the limit of weak noise and small nonlinearity a the mean first passage times and the escape rates. Based on the exact integral equation for the MFPT, (3.7), we found an *upper bound* on the MFPT, (3.8), which is solely determined by the noise probability.

If one confronts the results in this paper with those for time-homogeneous Fokker–Planck processes, one detects several characteristic differences. First we note that the MFPT exhibits, in contrast to the Fokker–Planck case, a jump [see (3.10)] at the exit boundaries, which in size does not exceed half of the maximal value of the MFPT. Moreover, one observes from the structure in (3.18) that the mean first passage time dynamics of a discrete metastable map requires additional information from the dynamics outside the domain of attraction. This is in clear distinction to the mean first passage time T^{FP} of a Fokker–Planck process, such as, e.g., the well-known result for the one-dimensional diffusive motion of a particle in a metastable potential well:

$$T^{\mathrm{FP}} = \left(\frac{\pi}{D(0)\,|U''(0)|}\right)^{1/2} \frac{\int_{\text{well}} W(x)\,dx}{W(0)}$$

where $D(x)$, $U(x)$, and $W(x)$ denote the diffusion coefficient, the potential, and the stationary probability, respectively, and where the barrier is located at $x = 0$.

An analytical treatment of the integral equation (3.4) for $t(x)$, or (3.15) for the boundary layer function $h(x)$, is characterized by several difficulties. Clearly, none of the integral equations occurring in Section 3 is of a standard type such as, e.g., the convolution type or the Wiener–Hopf type. Only due to the numerically established scaling relation (3.25) have we been able to evaluate the MFPT (at weak noise) explicitly.

In contrast, the rate approach in Section 4 is rather direct, whereby one bypasses some difficulties inherent with the integral equation for the MFPT. The result for the rate(s) in (4.11) and (4.12) renders the connection with the inverse mean first passage time, i.e., it equals the familiar "one over two times the MFPT" relationship well known from the Fokker–Planck case.[1,2,34,37,38]

We are attempting to generalize the theory for the MFPT to more general situations, such as, e.g., the hopping between two pairs of period-2 attractors. The evaluation of the invariant probability entering (3.18), as well as the solution of the boundary layer function $\tilde{h}(x)$, however, will be even more difficult.

APPENDIX. THE LIMIT OF CONTINUOUS TIME

Let us consider the multiple of a small time unit a in the limit $a \to 0$; i.e.,

$$na = t \qquad \text{for} \quad a \to 0, \quad n \to \infty \tag{A.1}$$

Then, the map

$$x_{n+1} = f(x_n) = x_n - aU'(x_n) \tag{A.2}$$

becomes for $a \to 0$ a continuous-time differential equation

$$\dot{x} = -U'(x) \tag{A.3}$$

In the case of a noisy map (1.2)

$$x_{n+1} = f(x_n) + \xi_n \tag{A.4}$$

with the probability for the noise amplitude [see (2.2)] given by

$$P[\xi_n \in (\xi, \xi + d\xi)] = (2\pi\varepsilon)^{-1/2} \exp[-\xi^2/(2\varepsilon)]\, d\xi \tag{A.5}$$

we consider the limit toward white Gaussian noise $\varepsilon \to 0$ such that

$$D = \varepsilon/a, \qquad \varepsilon \to 0, \quad a \to 0 \tag{A.6}$$

is kept a constant.

The generator Γ of the resulting stochastic process is determined from the backward equation

$$\Gamma^+ \phi(y) = \lim_{a \to 0} \frac{1}{a} \left[\int P_a(x \mid y)\, \phi(x)\, dx - \phi(y) \right] \tag{A.7}$$

where, from (2.3),

$$P_a(x \mid y) = \frac{1}{(2\pi D a)^{1/2}} \exp \frac{-[x - y + aU'(y)]^2}{2Da} \tag{A.8}$$

Using $u = (x - y)/\sqrt{a}$ as a new integration variable, one finds

$$\begin{aligned} \Gamma^+ \phi(y) &= \lim_{a \to 0} \frac{1}{a} \left\{ \int (2\pi D)^{-1/2} \exp \frac{-[u + a^{1/2} U'(y)]^2}{2D}\, \phi(y + a^{1/2} u)\, du - \phi(y) \right\} \\ &= \frac{D}{2} \phi''(y) - U'(y)\, \phi'(y) \end{aligned} \tag{A.9}$$

In other words, the continuous-time limit (A.1) and (A.6) induces the Fokker–Planck dynamics

$$\Gamma p_t(x) = +\frac{\partial}{\partial x} [U'(x)\, p_t(x)] + \frac{D}{2} \frac{\partial^2}{\partial x^2} p_t(x) \tag{A.10}$$

With the result (A.10) in mind, the current in (4.10) can be evaluated if we consider the limit

$$I^-(\hat{x}) = \lim_{a \to 0} \frac{1}{a} J_0^-(\hat{x}) \tag{A.11a}$$

where [see (4.3)]

$$J_0^-(\hat{x}) = \int_{-\infty}^{\hat{x}} dx \int_{\hat{x}}^{\infty} dy\, P(x \mid y)\, W_0^-(y) - \int_{\hat{x}}^{\infty} dx \int_{-\infty}^{\hat{x}} dy\, P(x \mid y)\, W_0^-(y) \tag{A.11b}$$

The corresponding result is easily read off from (A.10) as

$$I^{-}(\hat{x}) = U'(\hat{x})\, W_0^{-}(\hat{x}) + \tfrac{1}{2} D W_0^{-\prime}(\hat{x}) \tag{A.12}$$

At a stationary point, we have $U'(\hat{x}) = 0$; i.e., at $x_- = 0$

$$J^{-}(0) = aI^{-}(0) = \tfrac{1}{2}\varepsilon W_0^{-\prime}(0) = \tfrac{1}{2}\eta^{-\prime}(0)\, W(0) \tag{A.13}$$

because $W'(0) = 0$ at an extremal point.

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